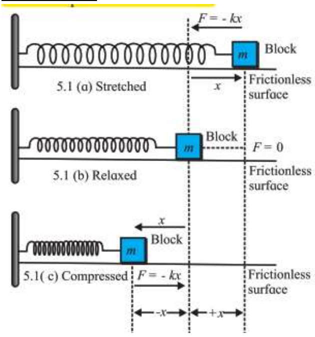
OSCILLATION

Definitions

Periodic Motion: Any motion which repeats after a definite interval of time
*NOTE: Every oscillatory motion is periodic but every periodic motion need not be oscillatory.

Linear SHM: is defined as the linear periodic motion of a body, in which force (or acceleration) is always directed towards the mean position and its magnitude is proportional to the displacement from the mean position

Linear SHM



The force in the spring is always directed to the mean position and is responsible to bring the block back. This restoring force is proportional to the displacement from the mean position.
f=-kx
where k:force constant, depends on the elastic properties of the spring.
x:displacement from the mean position.

-ve sign shows force is oppositely directed to the displacement.

$a = \frac{f}{m} = -\left(\frac{kx}{m}\right) = -\omega^2 x$, where m=mass.

NOTE: a α x and direction is opposite to displacement.
Force and acceleration both are directed towards the mean position.
At extreme positions, velocity=0, KE=0, magnitude of force is maximum (±kA) and magnitude of acceleration is maximum ($\pm\omega^2 A$)
At mean position, velocity magnitude is maximum (±ωA), KE=maximum, Force and acceleration=0
As the body moves from extreme position to mean position, the magnitude of displacement decreases, Force and acceleration magnitude decreases and velocity magnitude keeps increasing. On reaching the mean position, the KE is maximum, hence the body continues motion (even though the force & acceleration are zero) and the restoring force tends to decrease the magnitude of velocity till it comes to a stop (at the extreme position), where the force and acceleration magnitude becomes maximum and the body start acceleration in the direction of force (towards the mean position). This cycle keeps going on.

Differential Equation of Linear SHM

Force is proportional to displacement from the mean position and is directed towards the mean position
f=-kx where k: Force constant and x is displacement from mean position
By NSL, f=ma
Thus, ma = - kx
ma + kx =0

But, $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Thus, the differential equation of linear SHM is

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

ω: angular frequency = $\sqrt{\frac{k}{m}}$

Derive acceleration, velocity and Displacement using the Linear Differential Equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$a = -\omega^2 x$, Since $a = \frac{d^2x}{dt^2}$

This is the expression of acceleration in terms of displacement x.
At x=0, a_{min}=0, At x=±A, a_{max}=$\mp\omega^2 A$

Substitute 'a' as $\frac{dv}{dt}$ we get, $\frac{dv}{dt} = -\omega^2 x$

$$Thus, \frac{dv}{dx}\frac{dx}{dt} = -\omega^2 x$$

$$v\frac{dv}{dx} = -\omega^2 x \quad [Since\ \frac{dx}{dt} = v]$$

$$vdv = -\omega^2 x dx$$

$$Integrating\ both\ side, \int vdv = -\omega^2 \int xdx$$

$$\frac{v^2}{2} = -\omega^2\frac{x^2}{2} + c \ldots\ldots\ldots(i)$$

$$At\ x = \pm A, v = 0. Thus\ 0 = -\frac{\omega^2 A^2}{2} + c$$

$$Substitute\ c\ in\ (i) we\ get\ \frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 A^2}{2}$$

$$Therfore, v = \pm\omega\sqrt{A^2 - x^2}$$

This is the expression of velocity in terms of displacement from the mean position
At x=0, v_{max}=±ωA, At x=±A, v_{min}=0

Substitute 'v' as $\frac{dx}{dt}$ we get $\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$

$$Thus, \frac{dx}{\sqrt{A^2 - x^2}} = \omega dt$$

$$integrating\ both\ sides, \int\frac{dx}{\sqrt{A^2 - x^2}} = \omega\int dt$$

$sin^{-1}\left(\frac{x}{A}\right) = \omega t + \emptyset$, where ϕ is the initial phase or epoch
x=A sin(ωt+ϕ)
At t=0 if particle at mean position, t=0, x=0, $\emptyset = sin^{-1}\left(\frac{x}{A}\right)$= 0,π
At t=0 if particle at extreme position,

$$t = 0, x = -A, \emptyset = sin^{-1}(1) = \frac{\pi}{2}$$

$$t = 0, x = -A, \emptyset = sin^{-1}(-1) = \frac{3\pi}{2}$$

If particle starts from mean position, then at time t, x=A sin(ωt)
If particle starts from extreme position, then at time t
x= A sin(ωt+π/2) OR A sin(ωt - 3π/2) = ± Acos(ωt)

NOTE: As a function of time

$$x = Asin(\omega t + \phi)$$

$$v = \frac{dx}{dt} = A\omega\cos(\omega t + \phi)$$

$$a = \frac{dv}{dt} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 x$$

Amplitude, Period, Frequency, Phase:

Amplitude: The maximum displacement of a particle performing SHM from its mean position is called amplitude of SHM

Period*:* The time taken by the particle performing SHM to complete one oscillation is called period of SHM

$$x = Asin(\omega t + \phi)$$

$$At\ t = t + \frac{2\pi}{\omega}, x = Asin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right] = Asin(\omega t + 2\pi + \phi)$$

$$Thus, x = Asin(\omega t + \phi)$$

Hence, after $\frac{2\pi}{\omega}$ the particle is at the same place.

Thus after minimum time $\frac{2\pi}{\omega}$ the motion repeats.

Thus time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}$

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$$a = \omega^2 x, Thus\ T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{a}{x}}} = \frac{2\pi}{\sqrt{acceleration\ per\ unit\ displacement}}$$

Frequency: The number of oscillations per unit time performed by a particle performing SHM is called its frequency (n).

$$n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Phase*:* It gives the state of motion of a particle under motion (i.e. position and direction of motion). It is continuously changing with time.
Phase = θ = ωt + ϕ
Case 1: θ=0, indicates mean position and moving to the positive direction and during the beginning of the first oscillation. The particle will be at this same state at $(0+360)^o$ or $(0+2\pi)^c$ again i.e. beginning of the second oscillation and so on.
Case 2: θ=180^o or π^c, indicates particle at mean position moving to the negative direction. It will be in the same state in the second oscillation at θ=$(360+180)^o$ or $(2\pi + \pi)^c$ and so on
Case 3: θ=90^o or $\pi/2^c$, indicates particle is at the positive extreme position during the first oscillation. It will again be at the same state in the second oscillation when θ=$(360+90)^o$ or $(2\pi + \pi/2)^c$ and so on
Case 4: θ=270^o or $3\pi/2^c$, indicates particle is at the negative extreme position during the first oscillation. It will again be at the same state in the second oscillation when θ=$(360+270)^o$ or $(2\pi + 3\pi/2)^c$ and so on

Composition of two SHM having same period and same path

Consider a particle subjected simultaneously to two SHM's having the same period and along the same path but of different amplitude and different initial phase.

$x_1 = A_1 sin(\omega t + \phi_1)$ and $x_2 = A_2 sin(\omega t + \phi_2)$
The resultant displacement is given by $x = x_1 + x_2$
$Thus, x = A_1 sin(\omega t + \phi_1) + A_2 sin(\omega t + \phi_2)$
$= A_1 sin\omega t\ cos\phi_1 + A_1 cos\omega t\ sin\phi_1 + A_2 sin\omega t\ cos\phi_2 + A_2 cos\omega t\ sin\phi_2$
$= (A_1\ cos\phi_1 + A_2 cos\phi_2) sin\omega t + (A_1\ sin\phi_1 + A_2 sin\phi_2) cos\omega t$
$= Rcos\delta\ sin\omega t + Rsin\delta\ cos\omega t,$
$where\ Rcos\delta = A_1\ cos\phi_1 + A_2 cos\phi_2 \ldots .(i)$
$Rsin\delta = A_1\ sin\phi_1 + A_2 sin\phi_2\ \ldots ..(ii)$
$Therefore, x = Rsin(\omega t + \delta)$
This shows that the resultant of two SHMs of same period and same path is also an SHM with same period and resultant amplitude and initial phase as given below:

Resultant Amplitude $R = \sqrt{(Rsin\delta)^2 + (Rcos\delta)^2}$
Using (i) and (ii)

$$R = \sqrt{(A_1\ sin\phi_1 + A_2 sin\phi_2)^2 + (A_1\ cos\phi_1 + A_2 cos\phi_2)^2}$$

$$R = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2 sin\phi_1 sin\phi_2 + 2A_1A_2 cos\phi_1 cos\phi_2}$$

$$R = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}$$

$$(ii) \div (i) gives\ \frac{Rsin\delta}{Rcos\delta} = \frac{A_1\ sin\phi_1 + A_2 sin\phi_2}{A_1\ cos\phi_1 + A_2 cos\phi_2}$$

$$\delta = tan^{-1}\left(\frac{A_1\ sin\phi_1 + A_2 sin\phi_2}{A_1\ cos\phi_1 + A_2 cos\phi_2}\right)$$

Case 1: If $\phi_1 = \phi_2\ i.e. \phi_1 - \phi_2 = 0$

$$R = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2 cos0} = \pm(A_1 + A_2)$$

$If\ A_1 = A_2 = A, \quad R = 2A$

Case 2: If $\phi_1 - \phi_2 = 90^o\ out\ of\ phase$

$$R = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2 cos90} = \sqrt{{A_1}^2 + {A_2}^2}$$

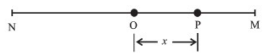
$If\ A_1 = A_2 = A, \quad R = \sqrt{2}A$

Case 3: If $\phi_1 - \phi_2 = 180^o\ out\ of\ phase$

$$R = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2 cos180} = |A_1 - A_2|$$

$If\ A_1 = A_2 = A, \quad R = 0$

Energy of a Particle Performing SHM:



Consider a particle of mass m performing linear SHM along path MN, with O being the mean position. At an instant let the particle be at P, which is x from O.
KE= ½ m v^2 = ½ $m\omega^2(A^2 - x^2)$= ½ $k(A^2 - x^2)$

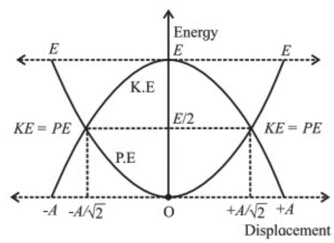
The restoring force at P is given by f = -kx , where k is force constant. Suppose particle is further displaced by a infinitesimal displacement dx against this force f, then the external work done
dW = f(-dx) = -kx(-dx)= kxdx
Total work done from O to P is given by

$$W = \int_0^x dW = \int_0^x kxdx = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$$

Total Energy E= $E_K + E_P$

$$E = ½\,k(A^2 - x^2) + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

Total Energy is a constant (not dependent on x and t). Hence energy is conserved in SHM.



Case 1: x=0, v_{max}= ωA,
E_{Kmax} = E = ½ kA^2 , E_{pmin} = 0

Case 2: x=±A, v_{min}=0, E_{Kmin}= 0
E_{pmax} = E = ½ kA^2

Case 3: KE= PE = E/2
½ $k(A^2 - x^2)$ = ½ kx^2 => $x = \frac{A}{\sqrt{2}}$

Case 4: x=±A/2,

$$E_P = \frac{1}{2}k\left[\left(\frac{A}{2}\right)^2\right] = \frac{kA^2}{8} = \frac{E}{4}, E_K = \frac{3E}{4}, Thus, E_K = 3E_P$$

$i.e. E_P = 25\%\ of\ TE\ and\ E_K\ is\ 75\%\ of\ TE\ at\ x = \pm A/2$

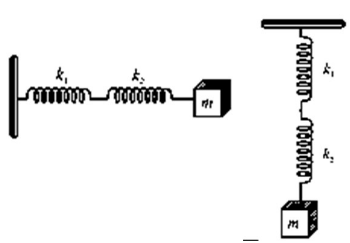
NOTE: $E = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}m(2\pi n)^2A^2 = 2\pi^2n^2A^2m = 2\pi^2\frac{A^2}{T^2}m$

Thus, TE is proportional to mass, square of amplitude, square of frequency and inversely proportional to square of Time period.

	Conical pendulum	Simple pendulum
1	Trajectory and the plane of the motion of the bob is a horizontal circle	Trajectory and the plane of motion of the bob is part of a vertical circle.
2	K.E. and gravitational P.E. are constant.	K.E. and gravitational P.E. are interconverted and their sum is conserved.
3	Horizontal component of the force due to tension is the necessary centripetal force (governing force).	Tangential component of the weight is the governing force for the energy conversions during the motion.
4	Period, $T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$	Period, $T = 2\pi\sqrt{\frac{L}{g}}$
5	String always makes a fixed angle with the horizontal and can never be horizontal.	With large amplitude, the string can be horizontal at some instances.

NOTE:

For springs attached in series f=-kx where $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$



For parallel springs f = –kx where $k=k_1+k_2$